Binomial Heap

Objective: In this lecture we discuss binomial heap, basic operations on a binomial heap such as insert, delete, extract-min, merge and decrease key followed by their asymptotic analysis, and also the relation of binomial heap with binomial co-efficients.

Motivation: Is there a data structure that supports operations insert, delete, extract-min, merge and decrease key efficiently. Classical min-heap incurs $O(n)$ for merge and $O(\log n)$ for the rest of the operations. Is it possible to perform merge in $O(\log n)$ time.

1 Binomial Tree

We shall begin our discussion with binomial trees. Further, we study structural properties of binomial trees in detail and its relation to binomial heaps. Binomial tree is recursively defined as follows;

1. A single node is a binomial tree, which is denoted as $B_0$
2. The binomial tree $B_k$ consists of two binomial trees $B_{k-1}$, $k \geq 1$.
3. Since we work with min binomial trees, when two $B_{k-1}$’s are combined to get one $B_k$, the $B_{k-1}$ having minimum value at the root will be the root of $B_k$, the other $B_{k-1}$ will become the child node.

Eg:

```
  5     -1
   |     |
  10   -5
     / \
    1   6
       /\  \
      7   2
```

Structural Properties:

For the binomial tree $B_k$,

1. There are $2^k$ nodes.
2. The height of the binomial tree is $k$.
3. There are exactly $\binom{k}{i}$ nodes at depth $i = 0, 1, \ldots, k$.
4. The root has degree $k$, which is greater than that of any other node, moreover if the children of the root are numbered from left to right by $k-1, k-2, \ldots, 0$, child $i$ is the root of the subtree $B_i$.

Note: Due to Property 3, it gets the name binomial tree (heap).

Eg:

```
  [5  -1  3  5  7  8  9]
```
Two $B_0$'s are merged to get a $B_1$.

Insert 3 into $B_1$, we get one $B_1$ and a $B_0$.

Insert 5, 3, 5, on merging two $B_0$'s → -1 3, on merging two $B_1$'s → -1 5

Insert 7, and 8. -1 7 8, on merging two $B_0$'s → -1 7 8, Insert 9, -1 7 9

Binomial Tree Construction

**Proof of Property 1:** Mathematical induction on $k$. The binomial tree $B_0$ is the base binomial tree for $k = 0$. Clearly, by definition, $B_0$ is a single node. Consider a binomial tree $B_k$, $k \geq 1$. Since $B_k$ is constructed using two copies of $B_{k-1}$, by the hypothesis, each $B_{k-1}$ has $2^{k-1}$ nodes. Thus, $B_k$ has $2^{k-1} + 2^{k-1} = 2^k$ nodes. Hence the claim.

**Proof of Property 2:** Mathematical induction on $k$. Clearly, $B_0$ has height '0'. By the hypothesis, $B_{k-1}$ has height $k - 1$. For $B_k$, one of the $B_{k-1}$'s becomes the root and hence the height increases by one when the other $B_{k-1}$ is attached. Thus, the height of $B_k$ is $k - 1 + 1 = k$.

**Proof of Property 3:** Let $D(k, i)$ be the number of nodes at depth $i$ for a binomial tree of degree $k$. Since $B_k$ is constructed using two copies of $B_{k-1}$, the nodes at depth $(i - 1)$ of $B_{k-1}$ becomes the nodes at depth $i$ for $B_k$. Therefore,

$$D(k, i) = D(k - 1, i) + D(k - 1, i - 1);$$

$$= \frac{(k - 1)!}{i!(k - 1 - i)!} + \frac{(k - 1)!}{(i - 1)!(k - 1 - i + 1)!};$$

$$= \frac{(k-1)!}{(k-i-1)!(i-1)!}\left[\frac{1}{i} + \frac{1}{k-i}\right];$$

$$= \frac{k!}{i!(k-i)!};$$

$$= \binom{k}{i};$$

**Proof of Property 4:** Follows from the recursive definition of $B_k$. 

---

2
2 Binomial Heap

In this section, we shall discuss the construction of min binomial heap, time-complexity analysis, and various operations that can be performed on a binomial heap along with its analysis.

A **Min Binomial Heap** $H$ is a collection of distinct min binomial trees. For each $k \geq 0$, there is at most one min binomial tree in $H$ whose root has degree $k$.

**Observation 1:** An $n$-node min binomial heap consists of at most $\lceil \log n \rceil + 1$ binomial trees.

**Observation 2:** A binomial heap on $n$ nodes and a binary representation of $n$ has a relation. Binary representation of $n$ requires $\lceil \log n \rceil + 1$ bits. Adding a node into a binomial heap $H$ is equivalent to adding a binary '1' to the binary representation of $H$.

We now present an example illustrating the construction of binomial heap and its relation to binary representation. For $B_i$, the value given in parenthesis is the binary representation of the number of nodes ($n = 2^i$) in $B_i$.

\[
\begin{array}{cccccccc}
-1 & 5 & 7 & 8 & 2 & 8 & 100 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
-1 & 5 & \rightarrow & 1 & 7 & 8 \\
B_0(1) & B_0(1) & & B_0(1) & B_0(1) & B_1(10) & B_1(10) & B_2(100) & B_0(1) & B_0(1) \\
\end{array}
\]

\[
\begin{array}{cccc}
-1 & 2 & 8 \\
7 & 5 & 8 \\
B_2(100) & B_0(1) & B_0(1) & n = B_2 + B_1 = 100 + 10 = 110 \\
\end{array}
\]

\[
\begin{array}{cccc}
-1 & 2 & 1 \\
7 & 5 & 8 \\
B_2(100) & B_1(10) & n = B_2 + B_1 + B_0 = 100 + 10 + 1 = 111 \\
\end{array}
\]

\[
\begin{array}{cccc}
-1 & 1 & 7 & 5 \\
7 & 5 & 2 & 100 \\
B_2 & B_2 & 2 & 100 \\
\rightarrow & 8 & 10 & 8 \\
B_2 & B_2 & 100 & 8 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 \ \\
1 & 0 & 0 \\
B_3 \\
\end{array}
\]
Note:

- In the above example, for \( n = 8 \), the final binomial heap has \( B_3 = 1 \) and \( B_2 = B_1 = B_0 = 0 \) which is 1000, the binary representation of 8.
- For \( n = 17 \), the binomial heap consists of one \( B_4 \) and \( B_0 \), which corresponds to the binary representation of 10001.

Insertion

Inserting a node into a binomial heap \( H \) is equivalent to adding a binary '1' to the binary representation of \( H \). In the worst case, the newly inserted node \( B_0 \) triggers merge at each iteration, i.e., inserting \( B_0 \) creates a new \( B_1 \) which in turn creates a new \( B_2 \) and so on. Thus, insert requires \( O(\log n) \) operations.

Merge

Merging two binomial heaps \( H_1 \) and \( H_2 \) is equivalent to adding two binary numbers. In particular, adding the binary representation of \( |H_1| \) and \( |H_2| \). In the worst case, every bit addition generates a carry which is equivalent to creating a new \( B_i \) while merging a copy of \( B_{i-1} \) in \( H_1 \) and a copy of \( B_{i-1} \) in \( H_2 \). Thus, merge incurs \( O(\log n) \) in the worst case, where \( n = |H_1| + |H_2| \). An example is illustrated below:

1. If \( B_0 \) is present in one of the heaps, then do nothing. Otherwise, merge two copies of \( B_0 \) and create one \( B_1 \). In general, merge two copies of \( B_i \) and create a copy of \( B_{i+1} \).
2. On merging we may get three copies of $B_1$, leave the first $B_1$ and merge the last two to obtain one $B_2$.

3. Now, two $B_2$ exists. Whenever, more than two copies of $B_i$ exists, leave the first one and merge the last two.

4. Now, merge two $B_3$.

There are 18 nodes, binary representation = $1 \ 0 \ 0 \ 1 \ 0$

**Extract Min**

Let $B_k$ be the node containing the minimum of a binomial heap $H$. By construction, $B_k$ contains $B_{k-1}, \ldots, B_0$ as its children. On extracting minimum, we invoke Merge() routine with $H_1$ being $B_{k-1}, \ldots, B_0$ and $H_2$ being the remaining nodes in $H$ (except $B_k$). Thus, extract minimum incurs $O(\log n)$ in the worst case. Suppose, we perform extract min on the above 18-node binomial heap, we get

After extracting $-7$,

Now we perform merge on the above binomial heap so that each $B_i$ occurs at most once.
Decrease Key

For decrease key, the value of the node pointed by the pointer \( x \) is decreased to the desired value \( y \). If \( y \) is smaller than its parent, i.e., on performing decrease key min binomial heap property is still maintained, then no further modification is required. Otherwise, min-heapify() routine is called to set right the min-heap property. Since the height of the binomial heap is \( k = \log n \), the decrease key in worst case takes \( O(\log n) \) comparisons.

Delete

To perform delete we make use decrease key and extract min subroutines. The node to be deleted is decreased to \( -\infty \) (or choose a value which is smaller than the current minimum), followed by extract min. Clearly, this incurs \( O(\log n) \).

Summary

In this lecture, we have discussed in detail a variant of min-heap, namely, min binomial heap using which one can perform the following operations efficiently.

- Insert and extract min can be done in \( O(\log n) \) time.
- Merging of two heaps can be done in \( O(\log n) \) in worst case, whereas classical heap incurs \( O(n) \).
- Decrease key and delete can be performed in \( O(\log n) \) time.

Acknowledgements: Lecture contents presented in this module and subsequent modules are based on the text books mentioned at the reference and most importantly, author has greatly learnt from lectures by algorithm exponents affiliated to IIT Madras/IMSc; Prof C. Pandu Rangan, Prof N.S.Narayanaswamy, Prof Venkatesh Raman, and Prof Anurag Mittal. Author sincerely acknowledges all of them. Special thanks to Teaching Assistants Mr.Renjith.P and Ms.Dhanalakshmi.S for their sincere and dedicated effort and making this scribe possible. Author has benefited a lot by teaching this course to senior undergraduate students and junior undergraduate students who have also contributed to this scribe in many ways. Author sincerely thank all of them.

References: