Discrete Mathematics - Problem Session -2 - All about FOL - Solutions.

1. We cannot help everyone, but everyone can help some one.
   $\forall x \exists y(\neg help(x,y)) \wedge \forall x \exists y(help(x,y))$
   $\forall x(\neg \forall y(help(x,y))) \wedge \forall x \exists y(help(x,y))$

2. No one who has no complete knowledge of himself will ever have a true understanding of another.
   This statement means, there exists one who has no knowledge of himeself will ever have a true understanding of another is false.
   $\neg[\exists x(\neg know(x,x) \wedge \exists y(understand(x,y)))]$
   $\forall x(\neg know(x,x) \rightarrow \forall y(\neg understand(x,y)))$

3. Negate the following: $\forall x \exists \epsilon((x > 0 \wedge \epsilon > 0) \wedge \forall y(y > 0 \rightarrow x - y \geq \epsilon))$
   `solution:` $\exists x \forall \epsilon((x > 0 \wedge \epsilon > 0) \rightarrow \exists y(y > 0 \wedge x - y < \epsilon))$
   NOTE: while applying negation, $\forall$ flips to $\exists$ and vice versa. Similarly, $\rightarrow$ flips to $\wedge$ and vice versa.

4. Some Republicans like all Democrats. No Republican likes any Socialist. Therefore, no Democrat is a Socialist. $\exists x(R(x) \wedge \forall y(D(y) \rightarrow like(x,y)))$ — (1)
   $\neg(\exists x(R(x) \wedge \exists y(S(y) \wedge like(x,y))) \equiv \forall x(R(x) \rightarrow \forall y(S(y) \rightarrow \neg like(x,y)))$ — (2)
   $\forall x(D(x) \rightarrow \neg S(x))$. — (3)

   The above claim is true, we shall present a direct proof.
   (3) EI,UI of (1) $R(a) \wedge (D(b) \rightarrow like(a,b))$, for some a and any b
   (4) UI of (2) $R(a) \rightarrow (S(b) \rightarrow \neg like(a,b))$, for any a and b
   Note: $P \wedge Q \rightarrow P$ is a tautology.
   (5) From (3), $R(a)$
   Note: $P \wedge (P \rightarrow Q)$ is a tautology.
   (6) From (5) and (4), $(S(b) \rightarrow \neg like(a,b))$
   (7) Contrapositive of (6); $like(a,b) \rightarrow \neg S(b)$
   Note: $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$ is a tautology.
   (8) From (3), we get $(D(b) \rightarrow like(a,b))$
   (9) From (8) and (7), we get $D(b) \rightarrow \neg S(b)$
   Since b is arbitrary,
   UG: $\forall x(D(x) \rightarrow \neg S(x))$, the required claim.

5. Prove or Disprove.
   $[\exists x P(x) \rightarrow \forall x Q(x)] \rightarrow \forall x[P(x) \rightarrow Q(x)]$
   . The Claim is true and we shall present below a proof.
   By definition; $\exists x P(x) \rightarrow \forall x Q(x) \equiv \neg \exists x P(x) \vee \forall x Q(x)$
   It follows that, $\forall x \neg P(x) \vee \forall x Q(x) \equiv \forall x(\neg P(x) \vee Q(x))$
   Thus, $\forall x(P(x) \rightarrow Q(x))$.

6. $\forall x[P(x) \rightarrow Q(x)] \rightarrow [\exists x P(x) \rightarrow \forall x Q(x)]$
   The above claim is false. We need a counter example which satisfies the premise and does not satisfy the conclusion. In particular, the example is such that $\exists x P(x)$ is true and $\forall x Q(x)$ is false.
   UOD: set of natural numbers, $P(x)$ x is divisible by 6, $Q(x)$ x is divisible by 2.
   It is clear that if a number is div by 6 then it is div by 2 and thus, $\forall x(P(x) \rightarrow Q(x))$ is true always.
   Also, $\exists x P(x)$ is true, for example $x = 6$.

Further, $\forall x Q(x)$ is false, not every natural number is divisible by 2.
Therefore, the conclusion is false.

7. Express the following using the first order logic by clearly mentioning UOD, predicates used: Everyone who gets admitted into an IIT gets a job. Therefore, if there are no jobs, then nobody gets admitted into any IIT.
Premise: $\forall x(person(x) \wedge (\exists y(IIT(y) \wedge admit(x,y))) \rightarrow (\exists z(job(z) \wedge getjob(x,z))))$
Conclusion: $\forall z(\neg(job(z)) \rightarrow \neg(\exists x(person(x) \wedge \exists y(IIT(y) \wedge admit(x,y))))$. (or)
$\neg\exists z((job(z)) \rightarrow \forall x(person(x) \wedge \exists y(IIT(y)) \rightarrow \neg admit(x,y))$.

The claim is false. Consider a venn diagram with two sets A : set of jobs B : students admitted in IITs. $A \wedge B$ denotes, students of some IIT gets job. $B \setminus A$ denotes students in other IITs.