DEPARTMENT OF SCIENCE AND HUMANITIES
(MATHEMATICS)
INDIAN INSTITUTE OF INFORMATION
TECHNOLOGY, DESIGN AND
MANUFACTURING KANCHEEPURAM
CHENNAI - 600127

Synopsis Of

Star Covers and Star Partitions of Graphs

A Thesis

To be submitted by

JOYASHREE MONDAL

For the award of the degree

Of

DOCTOR OF PHILOSOPHY

1 Abstract

A graph is called a *star* if it is isomorphic to the complete bipartite graph $K_{1,r}$ for some $r \geq 0$. When a graph models a network, like a road or computer network, each *induced* subgraph that is a star corresponds to a subnetwork that is a star network. The *center* of an induced star in such a graph potentially corresponds, in the underlying network, to either a bottleneck or a point that is desirable for locating some facilities; this is especially the case when the star has more than two vertices. Motivated by these practical considerations as well as their combinatorial appeal, we initiate the study two optimization problems, namely STAR COVER and STAR PARTITION, whose definitions appear below.

Consider any graph $G = (V, E)$. We also call any subset S of V a *star* of G if the subgraph induced by S, namely $G[S]$, is a star. A collection $\mathcal{S} = \{V_1, \ldots, V_k\}$ of stars of G is called a *star cover* of G if $V_1 \cup \ldots \cup V_k = V$. A star cover $\mathcal{S}$ of G is called a *star partition* of G if it is also a partition of V. The minimum k for which G admits a star cover of size k is called the *star cover number* of G and is denoted by $sc(G)$. The minimum k for which G admits a star partition of size k is called the *star partition number* of G and is denoted by $sp(G)$.

In our work, we study the following problems.

STAR COVER
Input: A graph G.
Output: A minimum star cover of G.

STAR PARTITION
Input: A graph G.
Output: A minimum star partition of G.

Both STAR COVER and STAR PARTITION are NP-hard in general. In this thesis, we obtain several structural, algorithmic and hardness results for both the problems restricted to many natural graph classes.

We obtain the following inequalities and identities involving the star cover number $sc(G)$ and the star partition number $sp(G)$. Here $\gamma(G)$ denotes the domination number and $\chi(G)$ denotes the chromatic number of a graph G. Also a graph H is called a butterfly graph if its vertex set partitions into a universal vertex and an induced $2K_2$. Each of these inequalities and identities has interesting algorithmic consequences.

- For any triangle-free graph G, $sp(G) = sc(G) = \gamma(G)$.
- For any butterfly-free graph G, $sp(G) = sc(G)$.
- For any co-disconnected graph G, $\chi(G)/2 \leq sc(G) \leq \chi(G)$.
- For any graph G, $\max\{\gamma(G), \chi(G)/2\} \leq sc(G) \leq \min\{sp(G), \gamma(G) \cdot \chi(G)\}$.

In this thesis, we also show that deciding whether an input graph can be covered by *or* partitioned into at most two stars has polynomial time algorithms but deciding whether an input graph can be covered by *or* partitioned into k stars is NP-complete for each fixed $k \geq 3$; these problems remain NP-complete even for K_4-free graphs. Consequently, we have that STAR COVER and STAR PARTITION are NP-hard, even for K_4-free graphs.

We prove that it is NP-hard to approximate STAR COVER and STAR PARTITION within $n^{1/2-\epsilon}$ for all $\epsilon > 0$, even for graphs of diameter two; see also Zuckerman (2007). Also both STAR COVER and STAR PARTITION do not have any polynomial time $c \log n$-approximation algorithm for some constant $c > 0$ unless P = NP; see Vazirani (2013).

From the literature on set partition (k-partition), it also follows that both problems have exact (a) $2^n n^{O(1)}$ time and exponential space algorithms and (b) $3^n n^{O(1)}$ time and polynomial space algorithms; see Björklund *et al.* (2009).

As already mentioned, we prove that $sp(G) = sc(G) = \gamma(G)$ for any triangle-free graph G, where $\gamma(G)$ denotes the domination number of G. In fact, our proof implies that the well-known DOMINATING SET problem is polynomially equivalent to each of STAR COVER and STAR PARTITION. Consequently, from the literature on DOMINATING SET, we have the following for both STAR COVER and STAR PARTITION: (1) The problems are NP-hard for (a) chordal bipartite graphs (see also Müller and Brandstädt (1987)), (b) $(C_4, C_6, \ldots, C_{2t})$-free bipartite graphs for every fixed $t \geq 2$ (see also Duginov (2014)) and (c) subcubic bipartite planar graphs (see also Garey and Johnson (1979)). (2) The problems have (a) $O(n^2)$ time $O(\log n)$-approximation algorithms for triangle-free graphs and (b) $O(n^2)$ time $(d+1)$-approximation algorithm for triangle-free graphs of degree at most d (see also Vazirani (2013)). (3) The problems have exact polynomial time algorithms for bipartite permutation graphs (see Brandstädt and Kratsch (1985); Farber and Keil (1985)), convex bipartite graphs (see Bang-Jensen *et al.* (1999); Damaschke *et al.* (1990)), doubly-convex bipartite graphs (see Bang-Jensen *et al.* (1999)) and trees (see Cockayne *et al.* (1975)). (4) With the number of stars in a star cover/partition as the parameter, from the work of Raman and Saurabh (2008), it follows that both the problems are W[2]-complete for bipartite graphs and are fixed parameter tractable for graphs of girth at least five (for an introduction to parameterized complexity one may refer to Downey and Fellows (2013)).

We also prove that both STAR COVER and STAR PARTITION are NP-hard for $K_{1,r}$-free graphs for each fixed $r \geq 3$ (in fact, the problems are NP-hard even for line graphs; see also Dor and Tarsi (1997)). Further, we present a polynomial time $\frac{r}{2}$-approximation algorithm for STAR PARTITION on $K_{1,r}$-free graphs (which implies a $\frac{3}{2}$-approximation algorithm for STAR PARTITION, for instance, on line graphs and cobipartite graphs as they are claw-free); see also Kelmans (1997). We also prove that both the problems are NP-hard for co-tripartite graphs, a subclass of $K_{1,4}$-free graphs; see also Maffray and Preissmann (1996).

For STAR COVER, we obtain a simple $O(n^2 t(n))$ time $O(\log n)$-approximation algorithm for any hereditary graph class for which the maximum independent set can be computed in time $O(t(n))$. Consequently, we have a polynomial time $O(\log n)$-approximation algorithm for STAR COVER on perfect graphs. We also observe that, for some $c > 0$, STAR COVER does not have a polynomial time $c \log n$-approximation algorithm for perfect graphs assuming P $\neq$ NP.

As already remarked, we prove that $sp(G) = sc(G)$ for any *butterfly-free* graph G. (A graph H is called a butterfly graph if its vertex set partitions into a universal vertex and an induced $2K_2$.) This result leads to $O(n^{14})$ time $O(\log n)$-approximation algorithms for both STAR COVER and STAR PARTITION on butterfly-free graphs. We also observe that, for some $c > 0$, neither of the problems has a polynomial time $c \log n$-approximation algorithm for butterfly-free graphs assuming P $\neq$ NP.

For both STAR COVER and STAR PARTITION, we have the following results on cographs and its subclasses: (1) $O(n^2)$ time exact algorithms for trivially perfect graphs ((C_4, P_4)-free graphs). (2) $O(n^2)$ time exact algorithms for co-trivially perfect graphs ($(2K_2, P_4)$-free graphs). (3) Linear time exact algorithms for threshold graphs ($(C_4, 2K_2, P_4)$-free graphs). For STAR COVER, we obtain a linear time 2-approximation algorithms for cographs.

For both STAR COVER and STAR PARTITION, we obtain the following results on split graphs: (1) The problems are NP-hard even for $K_{1,5}$-free split graphs for which the maximum degree of independent part vertices is three. (2) The problems have linear time exact algorithms for claw-free split graphs. (3) The problems have (a) linear time 2-approximation algorithms for split graphs and (b) $O(mn^{\frac{3}{2}})$ time $\frac{3}{2}$-approximation algorithms for $K_{1,4}$-free split graphs. We also obtain similar approximation algorithms for two superclasses of split graphs, namely split-like graphs and multisplit graphs.

Double-split graphs constitute an interesting class of perfect graphs and played an important role in the proof of Strong Perfect Graph Theorem; see Chudnovsky *et al.* (2006). We design $O(n^7)$ time exact algorithms for both STAR COVER and STAR PARTITION on double-split graphs. To show that the analyses of our algorithms are tight, we construct an intricate infinite family of double-split graphs with several properties. We also design a simple linear time algorithm for recognizing double-split graphs. Our work also suggests a useful succinct matrix representation for any double-split graph.

2 Objectives

- To understand the relation between star cover and star partition numbers $sc(G)$ and $sp(G)$ as well as their relation to other graph parameters when restricted to natural graph classes.
- To determine the computational complexity of both STAR COVER and STAR PARTITION on natural graph classes.
- To design good approximation algorithms and to prove matching inapproximabilty results for both the problems on natural graph classes.

3 Existing Gaps which were Bridged

DOMINATING SET and COLOURING are two well-known problems in the algorithms literature; see, for instance, Haynes *et al.* (1998) and Jensen and Toft (2011). Both STAR COVER and STAR PARTITION have some attributes of each of these classical problems. A star in a graph is essentially an *independent set* along with one exclusive vertex that *dominates* it. For each $1 \leq j \leq k$, suppose $Z_j = \{x_j\} \cup X_j$ is a star in a graph G with x_j as a center vertex. If $\mathcal{S} = \{Z_1, \ldots, Z_k\}$ is a star cover (parition) of G, then $\{x_1, \ldots, x_k\}$ is a dominating set of G. Also the independent parts $X_1, \ldots, X_k$ can be *assumed* disjoint even if $\mathcal{S}$ is a star cover of G by a result on star covers.

Kelmans (1997) considers the problem of covering a maximum number of vertices of the input graph by vertex-disjoint *induced* stars of the form $K_{1,i}, 1 \leq i \leq r$, where $r \geq 1$ is any fixed positive integer. He obtains a polynomial time algorithm for this problem.

The problem of partitioning (the vertex set of) an input graph into induced paths of length t ($t \geq 3$ is fixed) is NP-complete even for bipartite graphs with maximum degree three; see Monnot and Toulouse (2007). This result implies that the problem of partitioning a graph into induced $K_{1,2}$'s is NP-complete, even when restricted to bipartite graphs with maximum degree three.

The problem of partitioning an input graph into equal but fixed size stars, not necessarily induced, is investigated for many natural subclasses of perfect graphs in Van Bevern *et al.* (2017). (This problem is also referred to as STAR PARTITION in that paper.) This problem had already been shown to be NP-complete even when the star size is fixed to be three; see Kirkpatrick and Hell (1978).

Given a graph G and a positive integer k, the problem of deciding whether G can be partitioned into exactly k (not necessarily induced) stars, each of size at least two, is investigated by Andreatta *et al.* (2019).

Our work on STAR COVER and STAR PARTITION provides a bridge between the dominating set and vertex coloring problems. Our problems also fit into the context of the above-mentioned related problems.

4 Most Important Contributions

Bounds on STAR COVER and STAR PARTITION

- For any triangle-free graph G, $sp(G) = sc(G) = \gamma(G)$.
- For any butterfly-free graph G, $sp(G) = sc(G)$.
- For any co-disconnected graph G, $\chi(G)/2 \leq sc(G) \leq \chi(G)$.
- For any graph G, $\max\{\gamma(G), \chi(G)/2\} \leq sc(G) \leq \min\{sp(G), \gamma(G) \cdot \chi(G)\}$.

Deciding if a few Stars Suffice

- Deciding whether an input graph can be covered by *or* partitioned into at most two stars has polynomial time algorithms.
- Deciding whether an input graph can be covered by *or* partitioned into k stars is NP-complete for each fixed $k \geq 3$ even when restricted to K_4-free graphs.
- Consequently, STAR COVER and STAR PARTITION are NP-hard even for K_4-free graphs.

Inapproximability Results

- It is NP-hard to approximate STAR COVER and STAR PARTITION within $n^{1/2-\epsilon}$ for all $\epsilon > 0$ even when restricted to graphs of diameter two.

Triangle-free Graphs

- Both STAR COVER and STAR PARTITION are NP-hard for subcubic bipartite planar graphs.

$K_{1,r}$-free Graphs

- Both STAR COVER and STAR PARTITION are NP-hard for $K_{1,r}$-free graphs for any fixed $r \geq 3$ (in particular, for line graphs)
- STAR PARTITION has an $\frac{r}{2}$-approximation algorithm for $K_{1,r}$-free graphs (which implies a $\frac{3}{2}$-approximation algorithm for STAR PARTITION on line graphs and cobipartite graphs).
- Both STAR COVER and STAR PARTITION are NP-hard even for co-tripartite graphs (a subclass of $K_{1,4}$-free graphs).

Hereditary Graphs STAR COVER:

- Has an $O(n^2 t(n))$ time $O(\log n)$-approximation algorithm for any hereditary graph class for which the maximum independent set can be computed in $O(t(n))$ time.
- Has a polynomial time $O(\log n)$-approximation algorithm for perfect graphs and the $O(\log n)$ approximation factor cannot be significantly improved assuming P $\neq$ NP.

Butterfly-free Graphs

- Both STAR COVER and STAR PARTITION have $O(n^{14}m)$ time $O(\log n)$-approximation algorithms for butterfly-free graphs.
- The $O(\log n)$ approximation factor cannot be significantly improved assuming P $\neq$ NP.

Cographs

- Both STAR COVER and STAR PARTITION have:
  - $O(n^2)$ time exact algorithms for trivially perfect graphs ((C_4, P_4)-free graphs).
  - $O(n^2)$ time exact algorithms for complements of trivially perfect graphs ($(2K_2, P_4)$-free graphs).
  - Linear time exact algorithms for threshold graphs ($(C_4, 2K_2, P_4)$-free graphs).
- STAR COVER has linear time 2-approximation algorithms for cographs.

Split Graphs Both STAR COVER and STAR PARTITION:

- Are NP-hard even for $K_{1,5}$-free split graphs.
- Have linear time 2-approximation algorithms for split graphs.
- Have linear time exact algorithms for claw-free (i.e., $K_{1,3}$-free) split graphs.
- Have $O(mn^{\frac{3}{2}})$ time $\frac{3}{2}$-approximation algorithms for $K_{1,4}$-free split graphs.

Generalizations of Split Graphs

- STAR PARTITION has a linear time 2-approximation algorithm for *split-like graphs*.
- STAR COVER has an $O(n^2m)$ time $O(\log n)$-approximation algorithm for *split-like graphs*.
- Unless P = NP, neither STAR COVER nor STAR PARTITION has a polynomial time $c \log n$-approximation algorithm for some $c > 0$, even on *bipartite bisplit graphs*.
- STAR COVER has an $O(n^2)$ time $O(\log n)$-approximation algorithm for *multisplit graphs*.
- STAR PARTITION on an $O(n^2)$ time $O(\log n)$-approximation algorithm for *butterfly-free multisplit graphs*.

Double-Split Graphs

- Both STAR COVER and STAR PARTITION have $O(n^7)$ time exact algorithms for double-split graphs.
- Proving that the analyses of our algorithms are tight requires the construction of an intricate infinite family of double-split graphs with several properties.
- A simple linear time algorithm for recognizing double-split graphs.
- Introduction of a natural matrix representation for double-split graphs that leads to a succinct matrix representation for any double-split graph.

5 Conclusion

In this thesis, we have investigated the STAR COVER and STAR PARTITION problems from several perspectives. We identify interesting bounds on the star cover and star partition numbers of graphs. We determine the complexity of deciding whether an input graph can be covered by *or* partitioned into at most k stars, where $k \geq 1$ is fixed. We obtain both approximation algorithms and inapproximability results for both STAR COVER and STAR PARTITION. We also discuss the exact exponential time and space complexities of both the problems. Finally, we investigate the computational complexity of both STAR COVER and STAR PARTITION on several natural graph classes.

While both STAR COVER and STAR PARTITION are NP-hard for P_5-free graphs, their computational complexity remains open for P_4-free graphs, i.e., cographs. It appears likely that the problems are polynomial time solvable for cographs.

We have obtained a $\frac{3}{2}$-approximation algorithm for $K_{1,4}$-free split graphs. But the computational complexity of the problems on $K_{1,4}$-free split graphs remains open. Improving the trivial 2-approximation algorithms for split graphs also remains an interesting algorithmic challenge.

Our approach to STAR COVER and STAR PARTITION on double-split graphs appears to be fairly natural. Thus it would be interesting to see if the problems have other algorithmic solutions, especially with better than $O(n^7)$ time complexity.

It would also be of interest to identify other natural graph classes for which the problems are either NP-hard or have polynomial time exact or approximation algorithms. Indeed we have the computational complexity status of these problems open even for cobipartite graphs.

6 Organization of the Thesis

The proposed outline of the thesis is as follows:

Chapter 1: Introduction

Chapter 2: The Nature of Star Cover and Star Partition Problems

Chapter 3: Cographs and its Subclasses

Chapter 4: Split Graphs and its Generalizations

Chapter 5: Double-Split Graphs

Chapter 6: Conclusion

Appendix A: Double-Split Graphs

7 List of Publications

1. M.A. Shalu, S. Vijayakumar, T.P. Sandhya, J. Mondal, Induced star partition of graphs. Discrete Applied Mathematics 319 (2022), 81-91.

3. J. Mondal and S. Vijayakumar, Star covers and star partitions of cographs and butterfly-free graphs, CALDAM 2024. *Accepted.*

2. J. Mondal and S. Vijayakumar, Star covers and star partitions of double-split graphs, Journal of Combinatorial Optimization. *Accepted for Publication (Pending Minor Revisions).*

4. J. Mondal and S. Vijayakumar, Star covers and star partitions of split graphs and its generalizations. *Manuscript.*

References

1. **Andreatta, G.**, **C. De Francesco**, **L. De Giovanni**, and **P. Serafini** (2019). Star partitions on graphs. *Discrete Optimization*, **33**, 1–18, doi:10.1016/j.disopt.2019.01.002.

2. **Bang-Jensen, J.**, **J. Huang**, **G. MacGillivray**, and **A. Yeo** (1999). *Domination in Convex Bipartite and Convex-Round Graphs*. Technical report, University of Southern Denmark. doi:10.5555/870702.

3. **Björklund, A.**, **T. Husfeldt**, and **M. Koivisto** (2009). Set partitioning via inclusion-exclusion. *SIAM Journal on Computing*, **39**, 546–563, doi:10.1137/070683933.

4. **Brandstädt, A.** and **D. Kratsch** (1985). On the restriction of some NP-complete graph problems to permutation graphs. *In Proc. 5th Int. Conf. Fundamentals of Computation Theory, LNCS 199, 53-62*. Springer, doi:10.1007/BFb0028791.

5. **Chudnovsky, M.**, **N. Robertson**, **P. Seymour**, and **R. Thomas** (2006). The strong perfect graph theorem. *Annals of Mathematics*, **164**(1), 51–229, doi:10.4007/annals.2006.164.51.

6. **Cockayne, E.**, **S. Goodman**, and **S. Hedetniemi** (1975). A linear algorithm for the domination number of a tree. *Information Processing Letters*, **4**, 41–44, `doi:10.1016/0020-0190(75)90011-3`.

7. **Damaschke, P.**, **H. Müller**, and **D. Kratsch** (1990). Domination in convex and chordal bipartite graphs. *Information Processing Letters*, **36**, 231–236, `doi:10.1016/0020-0190(90)90147-P`.

8. **Dor, D.** and **M. Tarsi** (1997). Graph decomposition is NP-complete: A complete proof of holyer's conjecture. *SIAM Journal on Computing*, **26**, 1166–1187, `doi:10.1137/S0097539792229507`.

9. **Downey, R. G.** and **M. R. Fellows** (2013). *Fundamentals of parameterized complexity*. Springer, London, `doi:10.1007/978-1-4471-5559-1`.

10. **Duginov, O.** (2014). Partitioning the vertex set of a bipartite graph into complete bipartite subgraphs. *Discrete Mathematics & Theoretical Computer Science*, **16**, 203–214, `doi:10.46298/dmtcs.2090`.

11. **Farber, M.** and **J. M. Keil** (1985). Domination in permutation graphs. *Journal of Algorithms*, **6**, 309–321, `doi:10.1016/0196-6774(85)90001-X`.

12. **Garey, M. R.** and **D. S. Johnson** (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York.

13. **Haynes, T. W.**, **S. T. Hedetniemi**, and **P. J. Slater** (1998). *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc., New York.

14. **Jensen, T. R.** and **B. Toft** (2011). *Graph Coloring Problems*. John Wiley & Sons, Inc.

15. **Kelmans, A. K.** (1997). Optimal packing of induced stars in a graph. *Discrete Mathematics*, **173**, 97–127, `doi:10.1016/S0012-365X(96)00121-5`.

16. **Kirkpatrick, D. G.** and **P. Hell** (1978). On the completeness of a generalized matching problem. *In Proc. of 10th ACM symposium on Theory of Computing, STOC '78, 240-245*. `doi:10.1145/800133.804353`.

17. **Maffray, F.** and **M. Preissmann** (1996). On the NP-completeness of the k-colorability problem for triangle-free graphs. *Discrete Mathematics*, **162**, 313–317, `doi:10.1016/S0012-365X(97)89267-9`.

18. **Monnot, J.** and **S. Toulouse** (2007). The path partition problem and related problems in bipartite graphs. *Operations Research Letters*, **35**, 677–684, `doi:10.1016/j.orl.2006.12.004`.

19. **Müller, H.** and **A. Brandstädt** (1987). The NP-completeness of steiner tree and dominating set for chordal bipartite graphs. *Theoretical Computer Science*, **53**, 257–265, `doi:10.1016/0304-3975(87)90067-3`.

20. **Raman, V.** and **S. Saurabh** (2008). Short cycles make W-hard problems hard: FPT algorithms for W-hard problems in graphs with no short cycles. *Algorithmica*, **52**, 203–225, `doi:10.1007/s00453-007-9148-9`.

21. **Van Bevern, R.**, **R. Bredereck**, **L. Bulteau**, **J. Chen**, **V. Froese**, **R. Niedermeier**, and **G. J. Woeginger** (2017). Partitioning perfect graphs into stars. *Journal of Graph Theory*, **85**, 297–335, `doi:10.1002/jgt.22062`.

22. **Vazirani, V. V.** (2013). *Approximation Algorithms*. Springer, Berlin, Heidelberg.

23. **Zuckerman, D.** (2007). Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, **3**, 103–128, `doi:10.4086/toc.2007.v003a006`.